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Two-point closures and their applications: report on a workshop

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This international scientific workshop was organized in Lyon, France, from 10 to 12 May 2000. Its focus was 'Two-point closures and their applications', with the understanding that the analysis and design of such models requires expert knowledge coming from a wide range of areas in turbulence research, e.g. experiments, numerical simulations, asymptotic models, etc.

In the global challenge of turbulence modelling, two-point closures prove useful in many ways. Two-point correlations and spectra are useful measures of the distortion of the eddy structure of turbulence by stratification, large-scale strains, rotation, etc. In some cases, e.g. near boundaries, spectra can be drastically changed. In addition to the accurate characterization of turbulence, the explicit computation of two-point correlations or spectra shows how the internal dynamics of the various scales of motion are affected by such distortion, especially the cascade process on which the production/dissipation relationship depends. Distortion can be the cause of large departures from isotropic homogeneous turbulence, pulling turbulent flows far away from the local equilibrium that is often assumed. A rather weak departure can allow the use of linearized theories such as rapid distortion theory, for the applicability of which rational bounds may be estimated by comparisons with weakly nonlinear calculations. A different approach is necessary when dealing with larger departures, for instance due to growth of instabilities. In that case new physical or similarity arguments have to be employed to obtain a satisfactory description of the modification to the cascade process, which can even undergo reversal in the limit when three-dimensional turbulence becomes two-dimensional. Of course, significant changes in spectra have direct implications for one-point measures of turbulence – which can be explicitly derived by integration of two-point correlations – used in most industrial closure schemes. Such one-point models consequently need to be adapted when turbulence is strongly affected by distortion.

1. Fundamental aspects and motivation

Two-point statistical closures (the direct interaction approximation, or DIA, the eddy-damped quasi-normal Markovian, EDQNM, and the test field model, TFM) were initially developed mainly for the special case of homogeneous, isotropic turbulence during the ground-breaking studies of the 1960s and 1970s (see e.g. Kraichnan 1959, Orszag 1970, Herring & Kraichnan 1971 among many others), but have since been extended to some *anisotropic* and even *inhomogeneous* flows, areas in which work continues today (see Batchelor 1953, for an introduction to the spectral formalism

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in homogeneous turbulence). Although such models are aimed at strongly nonlinear turbulence, their mathematical structure is closely related to that of weakly nonlinear theories (see Cambon & Scott 1999). For example, the theory of weak turbulence (Benney & Saffman 1966, Zakhrov, L'vov & Falkovich 1992), which has recently been of considerable interest in the geophysical context, exhibits strong similarities with two-point closures, even though very few studies illustrating the connections between the two approaches have appeared to date. Thus, the case of anisotropic, incompressible, homogeneous turbulence subject to different anisotropizing influences, such as rotation or stratification, and of weakly compressible turbulence, even in the isotropic case, present challenges which are currently being addressed using both two-point techniques and asymptotic theories of weak turbulence. It is therefore important to extend the domain of applicability of two-point closures by incorporating results from linear theory (i.e. of rapid distortion type, using methods from stability theory) and weakly nonlinear analyses, results which include at least some aspects of the real dynamics of the flow. For instance, the *response tensor*, i.e. the operator which describes how a spectrum responds to a delta function 'kick', is a key concept in DIA (Kraichnan 1959). In cases of distorted turbulence, a natural first-order approximation of this operator is constructed from the Green's function of the full viscous rapid distortion theory (RDT) problem (in contrast to using merely the viscous part in the isotropic case). This Green's function can be incorporated in DIA or EDQNM theories, improving the modelling of nonlinear transfer terms, as illustrated by Holloway & Hendershot (1977) and Carnevale & Martin (1982) in two-dimensional turbulence with Rossby waves, and in three-dimensional stratified rotating turbulence (extensively discussed below). Another example is given by Kevlahan & Hunt (1997), who provided estimates of the limits of applicability of pure linear theory for turbulence subjected to irrotational strain.

In the case of inhomogeneous turbulence, RDT has provided good predictions for shear flows, or in the presence of solid boundaries. In that case, a separation of scales allows local expansions to be carried out, and expresses the action of the walls or the large scales upon the smaller-scale turbulent motions, which behave according to classical dynamics. Weak coupling can then be used to express the feedback from the Reynolds stress tensor to the mean flow in weakly inhomogeneous RDT.

Effective practical implementations of turbulence models have until now been limited to the one-point approach, which provides model relations between dissipation rate, kinetic energy and/or individual Reynolds stress components. These models do not give direct access to length scales, such as the integral length scales in different directions of space. A rough estimate such as $L \sim k^{3/2}/\varepsilon$, or even anisotropic estimates based on the full Reynolds stress tensor R_{ij} are questionable in shear flows (e.g. for characterizing streak-like structures), and in rotating stratified flows (pancake or columnar structures for instance). In contrast, these quantities are by-products of spectral models ranging from RDT to nonlinear two-point closures, the results of which compare well with direct numerical simulations (DNS) and experimental data. Furthermore, single-point closure predictions of the quantities which are directly computed (k–ε, k–L, R_{ij}–ε), are also questionable in the presence of a modified energy cascade with spectral imbalance, even if one restricts attention to the pure homogeneous incompressible case. These weaknesses appear no matter how sophisticated the single-point model considered, for instance in models of the dynamics of R_{ij} when the 'rapid' pressure–strain correlation for complex anisotropization processes is compared with RDT results, or via the ε-equation and 'slow' pressure–strain tensor compared to nonlinear two-point closures.

Since these quantities are a greatly affected by external distortion (as explained almost thirty years ago by Launder & Spalding 1972) in cases when spectra change significantly, one-point methods need to be adapted. By aiming at improving two-point modelling, one recognizes that, compared to one-point methods, they are intrinsically more realistic, describing more of the physics of turbulence, such as the continuum of different scales, and providing a correct treatment of pressure fluctuations (via the formalism of projection onto solenoidal modes in the incompressible case). The improvement of two-point models, interesting and useful in its own right, also has direct consequences for one-point kinematic and dynamical modelling hypotheses.

A further valuable consequence of the fact that two-point techniques can be used to describe different turbulence scales is that they prove useful in the construction of subgrid models for large-eddy simulations. But two-point modelling is by no means limited to this single application, important though it may be. There are also several practical problems in which two-point spectra are needed, for instance when distortion of the spectra has strong physical consequences (e.g. for wind loading of structures, as in the study by Mann 1994) or for diffusion calculations. This last point is illustrated in the recent development of low-cost kinematic simulations of turbulent velocity fields with the help of RDT (Nicolleau & Vassilicos 2000).

We should also point out that increasing computational power has recently made possible comparisons between the models and DNS, as was illustrated during the workshop in many of the presentations.

A total of thirty participants attended the workshop, held at Lyon from 10 to 12 May 2000, coming from Europe, the United States and Japan. Forty-five minute communications were given during sessions devoted to various aspects of two-point turbulence modelling, from fundamental theory to more applied matters. Turbulence subjected to external stable stratification and/or rotation was given special emphasis by many participants, and a significant amount of effort was devoted to the extension to inhomogeneous flow cases. The close relationship between two-point statistical models and weak turbulence theories was also discussed at length during the workshop.

In the following sections, we report on the topics developed by the speakers, having somewhat arbitrarily organized them into three thematic groups, although the speakers often aimed their talks at a wider subject area. Section 2 concerns research primarily devoted to wave-turbulence and two-point closure theories, with models that utilize a fully anisotropic and/or modal formalism. In § 3, we report on the intensive efforts put into modelling inhomogeneous flows using two-point quantities. Multiscale approaches and engineering-related models whose structure derives from two-points models are described in § 4. The last section discusses some key ideas pertaining to current advances in two-point models and some future research directions arising from this workshop.

2. Theories using anisotropic and modal descriptions

G. F. Carnevale [Scripps Institution, UC San Diego]† described how the eddy-damped quasi-normal Markovian (EDQNM) model can be derived from renormalized perturbation theory. In this aspect, weak wave or resonant wave interaction transport equations can be seen as a limiting form of the EDQNM closure. Carnevale also showed the specific role of quadratic invariants of the system equations, reflecting the

† Speakers at the workshop are cited their affiliation within square brackets.

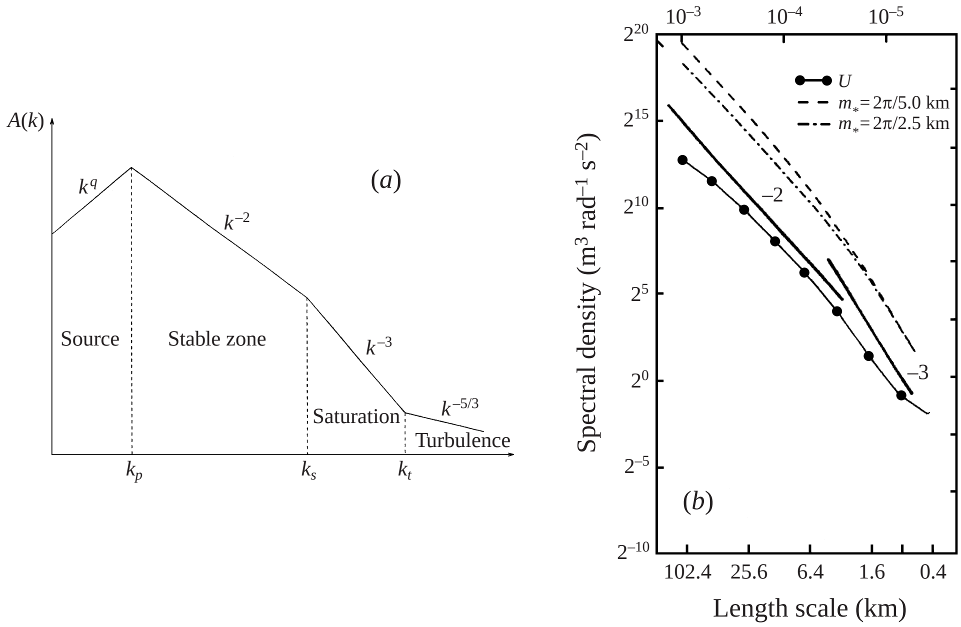

FIGURE 1. Results of Caillol for the energy spectrum of weak internal-wave turbulence: (*a*) scalings derived from an analytical model; (*b*) measurements by Bacmeister *et al.* (1996): horizontal wavenumber spectra of fluctuations of zonal velocity. The same two slopes are identified: −2 and −3. (*Courtesy P. Caillol and A. Zeitlin.*)

conservation properties of the base dynamical equations. For instance, conservation laws for the energy in resonant interactions bring to the fore three invariants of the initial systems of equations. One of these invariants appears in the asymptotic limit of wave turbulence but not in the underlying equations. Its conservation could suggest simple explanations or scaling laws for an anisotropic cascade process in more general three-dimensional cases. Potential applications, here restricted to the context of two-dimensional flows, include flows consisting of Rossby waves or internal gravity waves (Carnevale & Martin 1982; Carnevale & Frederiksen 1983).

The case of weakly interacting gravity waves was also the topic developed by P. Caillol [LMD, École Normale Supérieure]. He obtained evolution equations for the energy spectra, whether or not they include interactions with small mean potential vorticity (Caillol & Zeitlin 2000). Caillol showed that spectra become anisotropic in the vertical and horizontal (see figure 1), a feature also observed in computations of stably stratified turbulence modelled using anisotropic EDQNM (Godeferd & Cambon 1994). In both works, the slow dynamics of the vortex field is shown to lead to vertical collapse, i.e. the layering of the flow. Applications of this theory can be found in comparisons with existing atmospheric measurements, or measured oceanic gravity wave spectra, which show similar spectral scalings between Caillol's kinetic model equations, and atmospheric measurements.

In the different context of MHD, S. Nazarenko [University of Warwick] described a theory of weak triadic interaction of Alfvén waves. Both quasi-two-dimensional and isotropic spectrum models were derived, the latter obtained by averaging over the direction of the external magnetic field (Galtier *et al.* 2000). Quasi-two-dimensional states are obtained for weak turbulence subjected to a strong external magnetic field,

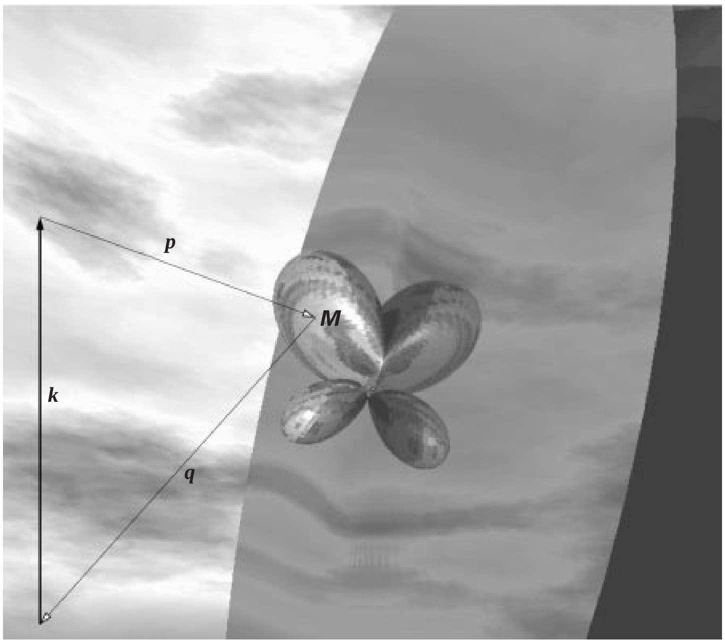

FIGURE 2. Sub-space, within the entire three-dimensional spectral space, for integration of the asymptotic model of inertial wave-turbulence by Scott. For each base wavevector $\boldsymbol{k}$ (here at an angle of 60° to the rotation axis), the sub-space is defined as the locus of points $\boldsymbol{M}$ where the triad of interacting wavevectors $\boldsymbol{k}$, $\boldsymbol{p}$ and $\boldsymbol{q}$ is resonant. (Note that the surface is drawn in a metallic finish over a cloudy sky; each component-surface corresponds to a different kind of resonant interaction with the infinite component (nearly flat) limited by the maximal wavenumber in the numerical implementation of the model.)

and, from this work, the dynamo effect appears simply as an equipartition of kinetic and magnetic wave energy.

The presentation by J. F. Scott [LMFA, École Centrale de Lyon] was devoted to the case of rotating turbulence. The research group in Lyon (see Cambon, Mansour & Godeferd 1997; Godeferd & Cambon 1994, and references therein) has developed an anisotropic EDQNM type model that includes the full linear wave solution, and takes advantage of the corresponding eigenmodes when modelling the nonlinear dynamics. An asymptotic model is found in the small Rossby number limit of the EDQNM model (Godeferd, Scott & Cambon 2000). The asymptotic model has the advantage that it requires triadic integration only on a sub-space of the entire spectral domain (a rendering of which is presented in figure 2, showing the complex topology of such surfaces), namely the resonant surfaces for inertial waves, thus greatly reducing computational costs, while retaining the essential anisotropizing effects of rotation on turbulence. A model including stratification and rotation is under development.

Regarding the combined effects of rotation and strain, O. Leuchter [ONERA] reported on a long-term study, which combines experimental (see figure 3), computational and theoretical aspects, including detailed comparisons of anisotropic EDQNM with DNS data. Emphasis was placed on a mean flow with elliptical streamlines, created by an original experimental set-up, and on rotating flow entering an axisymmetric convergent duct (Leuchter, Benoit & Cambon 1992). The two-point closure was shown to perform well for many rotating strained flows. As mentioned in § 1, this is due to the exact inclusion of the linear RDT operator, expressing effects

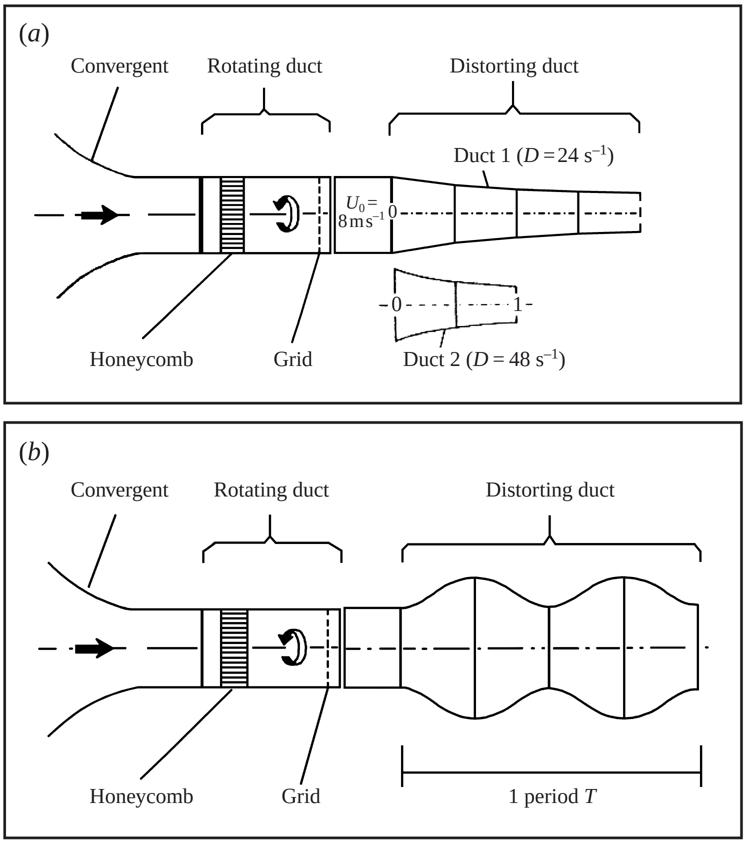

FIGURE 3. Experimental set-up of distorting ducts at ONERA, presented by Leuchter. (*a*) The axisymmetric contraction, (*b*) the elliptical streamline flow. (*Courtesy O. Leuchter*.)

of external distortion, and the corresponding modification of pressure–strain rate correlations. The latter terms can be extracted easily from the spectral equations, thus allowing comparison with one-point modelling.

The contribution by F. S. Godeferd [LMFA, École Centrale de Lyon] and C. Staquet [LEGI, Institut Polytechnique de Grenoble] also concerned the anisotropic EDQNM model, applied this time to the case of stably stratified turbulence. Comparisons with DNS allow them to show the validity of the model in predicting the anisotropic structuring of the flow at all scales, including the dissipative range (Staquet & Godeferd 1998; Godeferd & Staquet 2000). Good agreement with the model results lends confidence to the use of the model for much higher Reynolds number predictions than possible using DNS. Another future direction is the design of a subgrid-scale model adapted to the case of strongly stratified turbulence, in which the anisotropy at the level of the cut-off can be quantitatively characterized using EDQNM results. Considering the fact that the kinetic energy spectrum is shown to be two decades – a hundred times – larger for vertical wavenumbers than for horizontal ones, it is important that this anisotropy be reflected in a subgrid-scale model for large-eddy simulation (LES) of stably stratified turbulence. (Note that the anisotropy in the horizontal and vertical r.m.s. velocity components is only 50% in a typical case.)

Another model based on a Gaussian approximation, the direct interaction approximation (DIA), was presented by J. C. Hill [Iowa State University], and applied to the transport of active and passive scalars in homogeneous turbulent flows. Hill also showed comparisons with DNS, and mentioned the extension of the DIA code to the case of two scalars, e.g. salinity and temperature. The relevance of expansions in terms of spherical harmonics was extensively discussed for parameterizing detailed anisotropy in stably-stratified turbulence (Sanderson, Hill & Herring 1986).

One of the main pioneers of two-point models, J. R. Herring [NCAR, University of Colorado] reviewed the applications of the statistical theory of turbulence, namely two-point closures, to a variety of problems, among them: models of the far dissipation range of three-dimensional turbulence; two-dimensional decaying turbulence which raises the problem of the presence of coherent structures and intermittency; and the effects of waves, e.g. in stratified flows. In all these examples, recent DNS computations prove useful for assessing the validity of the model theory.

3. The challenge of inhomogeneous modelling

Even for homogeneous turbulence, going beyond the isotropic case entails a high computational cost for two-point simulations using classical closures, a cost which is not insignificant compared with that of direct, pseudo-spectral simulation. Thus, it is currently unattractive to solve the full set of equations resulting from closures such as DIA, TFM or EDQNM in the inhomogeneous case without simplifications. In this context, two types of approach seem particularly promising.

The first type of approach takes inhomogeneity into account via the basis set of modes used to express the fluctuations, while, as far as possible, maintaining the structure of equations of the correlation matrix similar to that of the homogeneous case. The modes which are substituted for Fourier components may, for instance, be chosen to satisfy the boundary and incompressibility conditions. This approach is illustrated by the recent work of L. Turner [LANL, Los Alamos], who considered the problem of channel flow using suitably chosen modes whose amplitude equations are analogous to those of Fourier modes in the homogeneous case and which are closed via a random phase approximation (Turner 1999). The normal modes of the linear problem might well be good candidates in this type of approach.

The second type of approach uses a dual, physical–spectral, representation of the two-point correlations in which Fourier decomposition with respect to the separation variable, $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}$, and a position variable such as $(\boldsymbol{x}' + \boldsymbol{x})/2$ are employed. The remaining necessary assumption is the separation of spectral and physical space dependences of the correlations, for example by treating the inhomogeneity as weak. Mathematically tractable techniques include using weighted spectral transforms (the Gabor transform, as in Nazarenko, Kevlahan & Dubrulle 1999 for rapid distortion theory), or methods suggested by short-wave linearized stability analysis (Lifschitz & Hameiri 1991). Outside linear theory, this approach is mainly illustrated by semi-empirical transport models, discussed below, which treat the dependence with respect to the position variable by analogy with one-point modelling.

As noted above, the high computational cost of full two-point modelling motivates attempts to find simplified models which include some of the more realistic physics of two-point closures, but which are less computationally onerous. Transport models for the joint physical–spectral space energy spectrum $E(k, \boldsymbol{x})$ have been developed, which describe inhomogeneity in a similar way to the diffusive terms in the k–ε model, but allow a better treatment of dissipation, calculated using a quasi-isotropic spectral model to describe the energy cascade. Such models lie somewhere between one-point and full two-point modelling in both cost and realism. Examples include the models

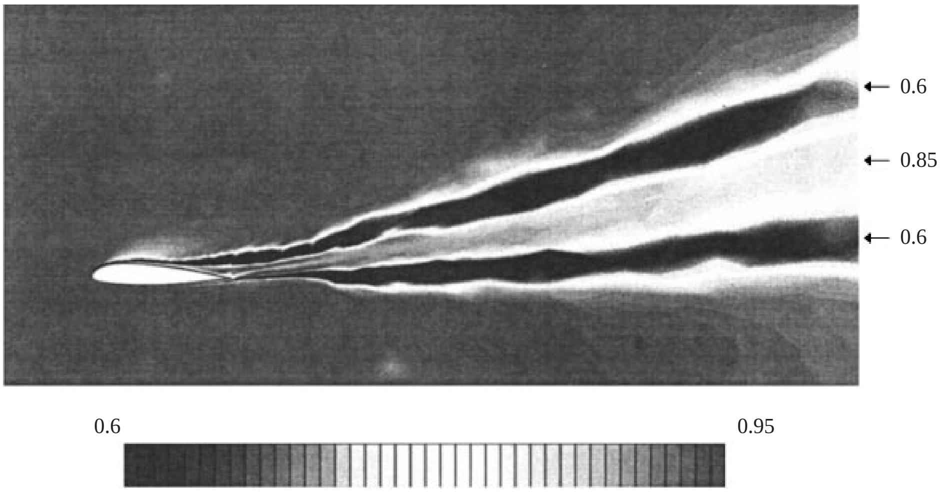

FIGURE 4. Application of the SCIT model to the flow over an F2A airfoil. The shading shows the ratio $\varepsilon_f/\varepsilon$ of large-scale energy flux to molecular dissipation, a measure of the lack of spectral equilibrium. (*Courtesy H. Touil & J.-P. Bertoglio.*)

of J.-P. Bertoglio and S. Parpais [ECL, Lyon] – the SCIT model, see below –, D. C. Besnard [CEA, Grenoble] (Besnard *et al.* 1996), T. Clark and C. Zemach [LANL, Los Alamos] and T. Burden [KTH, Stockholm].

J.-P. Bertoglio [LMFA, École Centrale de Lyon] presented two models. The first is an extension of the EDQNM model to account for the presence of weak inhomogeneity in one direction in an unbounded flow (Laporta & Bertoglio 1994). The model was shown to provide a good prediction of spatial transport of turbulent kinetic energy. The other approach, called the SCIT model (Touil, Bertoglio & Parpais 2000), is capable of treating wall-bounded flows, two-dimensional as of yet. It was used to compute the wake behind an airfoil (see figure 4) and the flow behind the valve of a piston engine, as an illustration of the different length scales that are present in such complex flows. In these applications, spectra are cut off at the larger scales when approaching the solid boundary, a phenomenon that is studied in detail by Hunt & Carlotti (2000) (see the following section).

The group at Los Alamos [LANL, Los Alamos] gave four talks devoted to the spectral modelling of inhomogeneous flows. The object of two of them was an application of EDQNM modelling ideas to a viscous channel flow bounded by two parallel free-slip plane walls. L. Turner presented the formalism in a highly compact form, with the use of a helicity decomposition (Turner 1999). He showed that a random phase approximation is a necessary hypothesis to confer some universality on the modelling approach. M. Ulitsky described the testing of a restricted random phase approximation, and validated it by performing DNS of the Navier–Stokes equations in the same configuration for fully developed turbulence without mean flow (Ulitsky, Clark & Turner 1999; Turner & Turner 2000). The statistics were examined by computing probability density functions (p.d.f.) of the modulus of normalized spectral covariance.

The other two talks dealt with the local wavenumber model of turbulence, which describes the evolution of turbulent spectra $E(\boldsymbol{x}, k, t)$ in time, as functions of both the physical space ($\boldsymbol{x}$) and the spectral space ($\boldsymbol{k}$) variables. T. Clark and C. Zemach showed comparisons of local wavenumber model predictions with results from experiments on homogeneous turbulence with irrotational strain and shear (Clark & Zemach 1992; Besnard *et al.* 1996). After the initial phase of relaxation toward equilibrium, the model is used to derive and determine the constants of one-point model equations of k–ε type. The problem of restoring, at least partially, anisotropy reflected as angular dependence of spectra in $\boldsymbol{k}$-space, was addressed by Zemach using a special basis of angular harmonics, whereas Clark presented the application of a numerical implementation of the local wavenumber model to the case of a free shear layer.

4. Other multiscale approaches

The talk by J. C. R. Hunt [University College, London] concerned the physical relationships between dynamics of eddy structures and the kinematics of spectra, especially the characterization of the eddy 'signature' rather than its energy or amplitude. As an application of these ideas, P. Carlotti [DAMTP, University of Cambridge] described how spectra of the different velocity components of a turbulent boundary layer can be modified by the presence of the wall, with a separation of regions into the 'surface layer' and the 'eddy surface layer', abbreviated as ESL (Hunt & Carlotti 2000). Defining the ESL thickness as the region in which a k^{-1} scaling is found for the horizontal velocity spectrum, this thickness appears to increase as the Reynolds number decreases, or when the surface roughness increases. In this zone the structure of turbulence contains many elongated horizontal eddies.

Two talks were devoted to the properties of small-scale structures. First, M. Larchevêque [LMM, Université Paris VI] gave a possible explanation of the $k^{-5/3}$ inertial range scaling for both incompressible and compressible turbulence, employing Lundgren's small-scale vortex in a compressible flow, with the help of two-dimensional DNS, in which shocklets were identified.

In the incompressible case, Y. Kaneda [Nagoya University] focused on non-universality of the small-scale statistics of turbulence. He discussed the anomalous scaling of small-scale anisotropy, presenting DNS at high resolution (1024^3), and identifying two classes of candidate flows that may explain the k^{-3} law of two-dimensional turbulence: forced turbulence, and another kind with non-trivial scaling in the enstrophy-cascade-range spectrum that depends on large-scale flow conditions.

Obtaining practical as well as accurate descriptions of complex flows was the guiding theme of two talks. P. Perrier [Cadas (formerly at Dassault Aviation)] stressed that the SCIT model (Bertoglio's presentation) was very good as a production/dissipation two-point model solver for flows with a wide range of scales, but still misses unsteady singular events, which need further modelling care; using data obtained from experimental particle image visualization Perrier showed that proper orthogonal decomposition helps identify such events.

P. Sagaut [ONERA, Paris] presented the concept of LES based on a dynamical *multiscale* closure for subgrid-scale modelling, an extension of the well-known dynamical subgrid scale model (Sagaut 1998). This amounts to a generalization of Germano's identity between different spectral bands, not necessarily adjacent, and expressed consistently with a multigrid type numerical scheme. Sagaut then presented a simulation for the mixing layer showing the validity of the extended model.

Two talks were devoted to the dispersion properties of anisotropic flows. J. C. Vassilicos [DAMTP, University of Cambridge] presented a Lagrangian model based on kinematic simulation and RDT that reproduces well the capping effect of stable stratification on one-particle vertical diffusion, and the two-stage levelling off in time for two-particle diffusion (Nicolleau & Vassilicos 2000). Y. Kimura [Nagoya

University] discussed the combined effects of stratification and rotation. He first showed how the structures in the turbulent field can become either pancake or cigar shaped, depending on the relative intensity of rotation and stratification (Kimura & Herring 1996). Kimura then presented preliminary results on single-particle dispersion, and the influence of stratification and rotation in modifying vertical dispersion.

R. Rubinstein [ICASE, NASA Langley] raised the problem of a unified description of transition and turbulence. He enumerated different starting points for a statistical description of the transition process – in terms of boundary layer receptivity, resonant interaction of Tollmien–Schlichting waves, or parabolized stability equations – each through evolution equations for the probability density of the mode amplitude.

5. Discussion and future directions

We now discuss the three main themes reported above, starting with the subject of wave-turbulence and two-point closure theories and models that take advantage of a fully anisotropic and/or modal description to capture more of the physics of turbulence.

The vast majority of previous research in these areas has concerned homogeneous isotropic turbulence. In the case of wave-turbulence, *statistical homogeneity* is one consequence of assuming random phases for the wave fields. Wave-turbulence theories have almost exclusively concentrated on isotropic power-law dispersion relations $\sigma = |\boldsymbol{k}|^\alpha$ when deriving Kolmogorov spectra, with the further key hypothesis of constant and isotropic energy flux across different scales associated with wavenumber $|\boldsymbol{k}|$ (Zakharov *et al.* 1992). By contrast, in geophysical flows, dispersion laws are anisotropic, with for instance $\sigma = \pm\beta k_x/k^2$ in the case of Rossby waves, $\sigma = \pm 2\Omega k_\parallel/k$ for inertial waves and $\sigma = \pm N k_\perp/k$ for gravity waves (k_x, $k_\parallel$ and $k_\perp$ are respectively the components of the wavevector in the zonal direction, and the directions parallel and perpendicular to the rotation/gravity vectors). In the latter two cases, anisotropy is reflected in the conical 'St Andrew cross' shape of isophase surfaces in typical experiments with a localized point forcing (see e.g. visualizations in Greenspan 1990; Godeferd & Lollini 1999), and by angular-dependent energy transfers when looking at nonlinear interactions (as illustrated by Godeferd & Staquet 2000; Godeferd *et al.* 2000). In the context of homogeneous turbulence subjected to uniform mean-velocity gradients, RDT solutions also exhibit strong anisotropy. In this case, RDT has the same analytical basis as an initial-value, linear stability analysis. As stressed during the workshop, the nonlinear problem of closure is best tackled by decomposing the background velocity and temperature fields in terms of the normal modes of the linear problem. Accordingly, nonlinear equations are written for the amplitudes of such modes or, more precisely, for their statistical correlations. The latter quantities are constant in the inviscid RDT limit. The correlation matrices are anisotropic, reflecting the spatial symmetries of the underlying dynamical equations. When divergence-free velocity fields are considered, the Craya–Herring frame of reference in three-dimensional Fourier space can be usefully employed (Craya 1958; Herring 1974), or equivalently a poloidal–toroidal decomposition in physical space, yielding a basis of solenoidal modes which facilitates the derivation of complete bases of eigenmodes (Cambon 2001).

Two-point closure and wave-turbulence theory have many common elements. Evolution equations for the mean spectral energy densities of waves are derived in wave-turbulence and are analogous to the spectral equations of homogeneous two-point closure. Time evolution is governed by similar energy transfer terms, cubic in the wave amplitudes (triads) (Benney & Saffman 1966). In some cases energy transfer involves fourth-order interactions (quartets) in wave-turbulence when triad resonances are forbidden by the dispersion law and/or by geometric constraints (e.g. shallow water waves, plasmas). However, when triple resonances are allowed, for instance in the cases of rotating, stably stratified and MHD turbulence addressed during the meeting, the wave-turbulence kinetic equations have the same structure as the corresponding equations of two-point closure in the limit of small interaction parameter (e.g. Rossby number, Froude number, magnetic Reynolds number in MHD) (e.g. Holloway & Hendershot 1977; Godeferd & Cambon 1994; Cambon *et al.* 1997). The precise form of the eddy damping parameter, which remains the heuristic correction to quasi-normal transfer in EDQNM, is unimportant in this limit (J. F. Scott, private communication). Its only role is to regularize the resonance operators, which reduce to Dirac delta functions in wave-turbulence. Moreover, provided care is taken, the 'Markovianization' process in EDQNM has similar consequences to the averaging procedure over the slow time of the wave-turbulence equations. The two approaches use closure hypotheses, namely Gaussian random phases in wave-turbulence and quasi-normality under EDQNM. Eddy damping, or more generally the nonlinear contribution to Kraichnan's response function, regains some importance for moderate interaction parameters, allowing extrapolation of wave-turbulence via two-point closure towards the case of strong interactions (e.g. purely isotropic turbulence without external or wave effects, for which classic two-point closure models are known to work satisfactorily). Proposals for renormalizing such generalized eddy damping were offered by G. Carnevale and R. Rubinstein during informal discussions.

Note that even if strict homogeneity is supposed (assuming it is permitted by the dynamical equations), the evolution equations resulting from two-point closure or from wave-turbulence may be complicated and computationally demanding. The main difficulty lies not in the number of independent spectra or co-spectra (multi-modal aspect), but in the angular dependence in Fourier space of these quantities. This dependence can be parameterized using angular harmonics, as shown by Zemach and Hill, but the number of harmonics needed becomes larger and larger as anisotropy develops. More generally, efficient numerical procedures have to be developed to render fully anisotropic two-point closure tractable, even in the homogeneous case. Interesting studies are in progress, such as the adaptation of pseudo-spectral schemes or Monte-Carlo techniques (Kaneda 1992). Another approach is to try to derive asymptotic models (Scott) and scaling laws (Caillol, Nazarenko) analytically, but simple arguments such as constant isotropic flux are totally inappropriate in this context. Some theoretical techniques do however exist for evaluating angular energy fluxes in the context of simplified anisotropic spectra such as $E \sim k_\perp^a k_\parallel^b$ for axisymmetric turbulence, as illustrated by Caillol & Zeitlin (2000) for gravity wave spectra, following ideas from Zakharov *et al.* (1992). It is also worth noting that a possible confusion arises between inhomogeneity and anisotropy, especially when considering vertically stratified flows. For instance, DNS and RDT results by Galmiche *et al.* (2001), which are presented as inhomogeneous, can be reinterpreted in the area of strictly homogeneous strongly anisotropic turbulence, if ensemble mean fields are treated as very low-frequency spectral contributions.

As regards two-point closure-based inhomogeneous flows, a variety of approaches are being developed. On the one hand, inhomogeneity resulting from solid boundaries may be accounted for in a 'rational' way, as illustrated by Turner (two-point closure for channel flows), and Carlotti (RDT for bounded shear flow). On the other hand, weak inhomogeneity far from boundaries can be tackled by ray techniques, or WKB approaches. It is worthwhile exploring how to perform the extension, to spectral turbulent transport equations, of asymptotic linear analyses in which disturbances represented in terms of wavepackets are convected and distorted following mean flow trajectories (Cambon & Scott 1999, § 5). In the presence of dispersive waves, it is also possible to advect weakly inhomogeneous turbulence spectra following the group velocity. Such applications of ray theories to turbulent transport have seen applications in stably stratified (J. Riley, F. S. Godeferd, private communications) and rotating (Le Penven, Bertoglio & Shao 1993) weak turbulence. Another example of an inhomogeneous generalization of EDQNM, incorporating modelling of triple velocity and pressure–velocity correlations, is the model by Laporta & Bertoglio (1994).

Ultimately, one wishes to derive engineering models using elements from two-point closure. Unfortunately, a full angular dependence in Fourier space is at present too complicated to account for in weakly inhomogeneous transport models of the spectral Reynolds stress tensor components $\Phi_{ij}(\boldsymbol{k}, \boldsymbol{x}, t)$. At most, spherically averaged spectra are considered, such as the kinetic energy spectrum $E(k, \boldsymbol{x}, t)$. Accordingly, one has to forget the idea of recovering the asymptotic RDT limit exactly, and needs to model the 'rapid' terms comprising distortion and pressure–strain correlations, modelling which is unnecessary in the fully anisotropic theory. This closure problem arises from use of the averaged equations and is treated similarly to single-point closure in both the local wavenumber approximation (Los Alamos group) and in the SCIT models (Touil *et al.* 2000). Alternative, more sophisticated procedures have been proposed by Cambon, Jeandel & Mathieu (1981), Reynolds & Kassinos (1995) and Cambon & Scott (1999), but are at present limited to homogeneous turbulence. In the same way, to date only heuristic corrections have been proposed for treating near-wall effects. Accordingly, the main advantage compared to classic single-point closure models is the allowance made for radial energy transfer, reflecting the 'classic' cascade, and resulting in a better prediction of the dissipation rate. For instance, the imbalance between radial energy flux from the largest scales, ε_f, and the dissipation rate ε is allowed for, as illustrated in figure 4.

During the meeting, other multiscale approaches were discussed in the context of LES (Sagaut), implicitly homogenized computational methods (Perrier), and the use of simplified eddy structure models (Larchevêque). As stressed by Perrier, proper orthogonal decomposition (Lumley 1967) modes are good candidates for describing strongly inhomogeneous flows using two-point stochastic models based on amplitudes obtained by Galerkin projection of the underlying dynamical equations. Since a small number of modes is sufficient to represent the energy-containing domain, low-dimensional dynamical models can be derived. These procedures cannot, of course, model the dynamically significant smallest scales, which terminate in the cascade process at the dissipative range. There remains work to do in matching low-dimensional very-large-scale dynamic models, for instance using proper orthogonal decomposition modes, with quasi-homogeneous two-point closure and subgrid-scale models.

Another area of interest is the Lagrangian statistics and dispersion of particles, as discussed by Vassilicos and Kimura, with important previous work using two-point closure, such as Lagrangian renormalized approximation, described by Kaneda (1992). The starting point of kinematic simulation, used by Vassilicos, is similar to that often used for initializing pseudo-spectral DNS, and the computation of trajectories using frozen velocity fields yields interesting results for one- and two-particle dispersion. Linear wave dynamics, as in RDT, can be incorporated very easily in the construction of the velocity field, including random phases and random orientation of wavevectors, and yielding realistic anisotropic dispersion in the presence of stratification, rotation, and no doubt other organizing effects such as mean shear. Note that RDT cannot directly be used to provide Lagrangian correlations due to the intrinsic nonlinearity of the equations of particle motion, even if the velocity field is governed by linear equations, but it still forms a crucial ingredient of Lagrangian nonlocal models (see also Kaneda & Ishida 2000), which include much more of the real dynamics than classic 'local' Lagrangian stochastic models. A complete review of multiscale approaches ought to include shell models, but this topic was not addressed during the meeting; we hope to incorporate them within the scope of the next workshop.

From the point of view of the scientific community, this workshop has permitted the identification of an international group of workers (mentioned throughout the preceding text) currently active in the area of two-point modelling. The meeting is expected to be the first of a series, with a second one likely to be hosted in the USA with Dr R. Rubinstein as coordinator. In the European context, a Special Interest Group (SIG) of ERCOFTAC is being launched, entitled 'Multipoint turbulence structure and modelling'.

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